CONTINUITY BETWEEN FINITE AND INFINITE ELEMENTS ALONG ARTIFICIAL BOUNDARY IN SUBSTRUCTURAL MODELS

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ABSTRACT

This paper describes and discusses some aspects of $C_0$ and $C_1$ continuity along finite/infinite element line in two-dimensional substructure soil-structure interaction problems. In this class of problems the finite/infinite element line marks the artificial

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boundary between the near and the far field of the model. The rate of the continuity has a direct effect on the accuracy of the computational results in wave propagation class problems. In addition the rate of the continuity affects directly the choice of the distance to the artificial boundary. However, the choice of h-version FEM or p-version FEM approach has significant importance for this class of problems. The analysis focuses on wave propagation type problems only. Several general remarks for the \( C_0 \) and \( C_1 \) continuity in the three dimensional soil-structure interaction models are reviewed.

1. Introduction

A large number of research papers in the late 1970s and in the 1980s gave the framework of infinite element formulations and related to different contributions. The basic contributions and new directions are referred to papers by Bettess [31, 34, 36] and Resende [37] and the book on infinite elements by Bettess [30]. The work of Bettess [31] introduced infinite elements in literature. This original contribution gives an infinite element, derived for a one dimensional problem and radial shape function defined by:

\[
W_i(\xi) = \exp\left(\frac{\xi - \xi_{i-1}}{L}\right) L_i(\xi), \quad 1 \leq i \leq N, \tag{1}
\]

where \( L_i(\xi) \) is the standard Lagrange interpolation polynomial at the node \( i \) located at a finite radial distance, and \( L \) is a positive parameter that can be arbitrarily chosen. The accuracy of the results depends on the \( L \). The decay term in equation (1) assures that products of shape functions and derivatives are integrable in a radial direction and numerical results can be obtained. This infinite element for the Laplace equation was extended to Helmholtz equation by Bettess and Zienkiewicz [45]. By inclusion of a complex valued factor equation (1) is modified as:

\[
W_i(\xi) = \exp\left(jk\xi\right) \exp\left(\frac{\xi - \xi_{i-1}}{L}\right) L_i(\xi), \quad 1 \leq i \leq N. \tag{2}
\]

The variational formulation used does not involve complex conjugated shape functions or weighting test functions. Indeed, such a weighting is not necessary to make radial integrals well defined due to the exponential terms in (2). The radial shape functions in (1) and (2) are not consistent with the radial behavior of the solution derived from the separation of variables. So this infinite element can only give some idea of the solution in the exterior domain. Chow and Smith use shape functions similar to (2), but they replace the Lagrange interpolation polynomial with so called Serendipity family of polynomials.

The work of Lynn and Hadid [38] proposes the radial shape function in the form:

\[
W_i(\xi) = \xi^{-i}, \quad 1 \leq i \leq N. \tag{3}
\]

The authors actually used a linear combination of the decay function in their definition. Obviously (3) is defined consistently with the decay rate and therefore such radial shape functions allow to correct representation of the solution in the exterior domain.

Medina in [39] described an infinite element formulation similar to the previous one and noted that the asymptotical behavior of the solution has the leading order term \( \xi^{-1} \).
The work of Zienkiewicz, Emson and Betess [40] improved the mapped infinite element formulations. They introduced a mapping in the form:

\[ x = -\frac{\xi}{1-\xi} x_0 + \left(1 + \frac{\xi}{1-\xi}\right) x_2, \]  

(4)

which maps the interval \( \xi \in [-1,1] \) to \( x \in [x_1, x_3] \) where \( x_3 = \infty \). The point \( x_2 \) is a midpoint in the reference interval \( x \in [x_1, x_3] \) and \( x_0 \) is given such that, \( x_1 \) lies midway between \( x_0 \) and \( x_2 \). The polynomials defined on the reference element are transformed by this mapping to functions with reciprocal power of \( \xi \). A polynomial \( p(\xi) \) can be written as

\[ p(\xi) = \sum_{j=1}^{N} \alpha_j \xi^j = \sum_{j=1}^{N} \beta_j \tilde{\xi}^{-j}, \]  

(5)

where \( \tilde{\xi} = x - x_0 \) is a distance between the point O ("pole") and a general point on \( x \) direction.

It is obvious from Eq. (4) that when:

\[ \xi = 0 \rightarrow x = x_2, \]
\[ \xi = 1 \rightarrow x = \infty, \]  

(6)

\[ \xi = -1 \rightarrow x = \frac{x_0 + x_2}{2} = x_1. \]

Finally the mapping can be expressed as:

\[ x = \left(1 + \frac{2\xi}{1-\xi}\right) x_2 - \frac{2\xi}{1-\xi} x_1 = N_2 x_2 + N_1 x_1. \]  

(7)

Definitely, the standard polynomial shape functions of order \( N \) represent the \( N \) leading order decay rates. Therefore, this mapping of infinite element leads to a formulation which can consistently represent the solution in the exterior domain.

Astley and Eversman [42], [43] and [44] solve the exterior Helmholtz problem and analyze what they call wave envelope element. This is basically an infinite element in which the radial shape function used is of the form:

\[ W_r(\xi) = \xi^{-1} \exp(ik\xi). \]  

(8)

### 2. Recent infinite element developments

The basic idea of some recent research is to couple the advantages of the finite element method with some infinite element approximation techniques to simulate more properly wave propagation in elastic media. This simulation is very important for a wide class of problems in computational mechanics. In structural mechanics, for instance, this are soil-structure interaction problems. For getting adequate results in soil-structure interaction
problems it is very important to consider the energy radiation due to the vibration of the structure into the far field medium. These SSI effects are very often important especially in the case of stiff and massive structures resting on relatively soft ground. In static analysis, the simple truncation of the far field with setting of appropriate boundary conditions very often gives good results. However, in dynamic cases, an artificial boundary made by truncation makes results erroneous because of the reflection of the waves.

In the last decades, much work has been done on unbounded domain problems and several types of modeling techniques have been developed to avoid these effects. Such techniques are: the viscous boundary, transmitting boundary, boundary elements, infinite elements and system identification method. At the same time, several numerical methods for these types of problems have been suggested. The basic idea of these approaches is to divide domain $\Omega$ into two parts, the bounded part $\Omega_c$ and unbounded part $\Omega_\infty$, where for the first one $x_i \leq \xi_j$. For more appropriate simulations we need to imply the assumption $u(x_j) = 0$ on $\Omega_\infty$.

Considering these approaches, using infinite elements is a good way to solve soil-structure interaction problems, since its concept and formulation are much close to those of the finite element method except for the infinite extent of the element domain and the shape function in one direction. In this case, there is no loss of symmetry of the element matrices. The unbounded domain $\Omega_\infty$ is partitioned into a finite number of infinite elements directly incorporated with the element mesh in the bounded domain $\Omega_c$. In the numerical models these domains are very often called near ($\Omega_c$) and far ($\Omega_\infty$) fields. The idea for an artificial boundary between the near and the far fields is shown in Fig. 1.

![Fig. 1. Artificial boundary between the near field and the far field](image)

This paper is a continuation of the work [27] and proposes some new developments of the formulation of the normalized superposed elastodynamical infinite element (NSEIE) suggested recently in the same work.
3. Isotropic normalized superposed elastodynamical infinite element (NSEIE)

The displacement field in the elastodynamical infinite element can be described in the standard form of the shape functions based on wave propagation functions [27] as:

\[ u(x, z, \omega) = \sum_{i=1}^{n} \sum_{q=1}^{m} N_{iq} (x, z, \omega) p_{iq}(\omega), \quad \text{or} \quad u(x, z, \omega) = N_{p}(x, z, \omega) p(\omega), \]  

(9)

where \( N_{iq}(x, z, \omega) \) are the standard shape displacement functions, \( p_{iq}(\omega) \) is the generalized coordinate associated with \( N_{iq}(x, z, \omega) \), \( n \) is the number of nodes for the element, and \( m \) is the number of wave functions included in the formulation of the infinite element.

For horizontal wave propagation the basic shape functions for the HIE type infinite element can be expressed as:

\[ N_{iq}(x, z, \omega) = L_{i}(\eta) W_{q}(\xi, \omega), \]  

(10)

where \( W_{q}(\xi, \omega) \) are horizontal wave functions based on the theory of wave propagation in elastic isotropic media with an infinite size of one direction.

By taking into account only the real parts of the wave functions the equations of the wave propagation can be written as follows:

\[ \text{Re} W_{q}(\xi, \omega) = \cos \left( \frac{i\omega}{c_{s}} \xi \right) e^{-\alpha_{s} \xi}, \quad \text{or} \quad \text{Re} W_{q}(\xi, \omega) = \cos \left( \frac{i\omega}{c_{p}} \xi \right) e^{-\alpha_{p} \xi}, \]  

(11)

where \( c_{s}, c_{p} \) are the wave velocities for \( S \)-waves and \( P \)-waves respectively.

Expanding these functions in a trigonometric or Fourier like series for all wave functions included in the formulation of the infinite element, the shape functions for HIE can be written in the form:

\[ \text{Re} W_{q}(\xi, \omega) = \sum_{n=-\infty}^{\infty} A_{n} \cos \left( \frac{im\omega}{c_{s}} \xi \right) e^{-\alpha_{s} \xi}, \quad \text{or} \quad \text{Re} W_{q}(\xi, \omega) = \sum_{n=-\infty}^{\infty} A_{n} \cos \left( \frac{im\omega}{c_{p}} \xi \right) e^{-\alpha_{p} \xi}, \]  

(12)

where \( \omega_{0} \) is the first frequency and \( \omega = n\omega_{0} \).

The coefficients \( A_{n} \) can be written as:

\[ A_{n} = \int_{0}^{T} \text{Re} W_{q}(\xi, \omega) \cos \left( \frac{in\omega_{0}}{c_{s}} \xi \right) d\omega. \]  

(13)

Because \( m \) is a finite number and \( A_{n} = 1 \) can be used for the shape function, the formulation of Eq. (12) can be expressed as:

\[ \text{Re} W(\xi) = \frac{1}{m} \sum_{q=1}^{m} \sum_{n=-\infty}^{\infty} \cos \left( \frac{im\omega_{0}}{c_{s}} \xi \right) e^{-\alpha_{s} \xi}, \quad \text{or} \quad \text{Re} W(\xi) = \frac{1}{m} \sum_{q=1}^{m} \sum_{n=-\infty}^{\infty} \cos \left( \frac{im\omega_{0}}{c_{p}} \xi \right) e^{-\alpha_{p} \xi}. \]  

(14)

Using this approach and superposing on the \( m \) components, we finally obtain:

\[ N_{i}(x, z) = \sum_{q=1}^{m} N_{iq}(x, z, \omega) = L_{i}(\eta) \text{Re} W(\xi, \omega). \]  

(15)
Equation (15) is the so-called united shape function for the infinite element based on the finite number of wave propagation functions.

Each shape function $N_{iq}(x, z, \omega)$ is associated with only one frequency and $p_{iq}(\omega)$ is a generalized coordinate involving a single wave component only. Then the component matrices $k_{iq}$ and $m_{iq}$ can be presented as

$$ k_{iq} = \int_{\Omega} \overline{B}_i^T D\overline{B}_i d\Omega, \quad \text{and} \quad m_{iq} = \int_{\Omega} \overline{N}_i^T \overline{N}_i d\Omega I, \quad (16) $$

where $\overline{B}_i = [\partial](\overline{N}_i) = [\partial](LW)$ and $[\partial]$ is a linear differential operator matrix.

The assembly of component matrices in the element matrix is made in the standard and well known way.

### 4. New approximation functions form

In the last few years, a lot of efforts have been focused on was expended on a finding the most suitable approximation functions for the elastodynamical infinite elements [18], [21] and [23]. The shape function given by Eq. (10), suggested in an exponential form in [19], is very effective only in the cases when the artificial boundary lies away from the structure. Such a form ensures only $C^0$ continuity through $x_b$ line (boundary between finite elements and infinite elements) and between infinite element-infinite element lines, Fig. 1.

This paper suggests a new form of the shape function, which can be expressed as:

$$ N_{iq}(x, z, \omega) = L_i(\eta)W_{q}(\xi, \omega), \quad (17) $$

where $\eta \in [-1, 1]$ and $\xi \in [0, \infty]$.

The first member on the right hand side is the same Lagrange polynomial as in equation (3) but we will express the second one as:

$$ N_{iq}^{fd}(x, z, \omega) = L_i(\eta)W_{q}^{fd}(\xi, \omega), \quad (18) $$

for the nodes on the line $\xi = 0$ or $x_b$ line in the global coordinate system and

$$ N_{iq}^{fd}(x, z, \omega) = L_i(\eta)W_{q}^{fd}(\xi, \omega), \quad (19) $$

for the nodes in $\xi = \infty$.

![Fig. 2. Scheme of the boundary between finite and infinite elements (adjacent finite and infinite element)](image)
5. Continuity along finite and infinite elements

The continuity through finite and infinite elements can be enforced in exactly the same way as between two finite elements because they have the same degrees of freedom and approximation polynomial degrees. The degree of the approximation polynomial in finite or infinite element formulation depends directly on the differential degree.

As mentioned above, the exponential form of the shape functions ensures only $C^0$ continuity through boundary between a finite element and an infinite element. However, the use of Eq. (18) and Eq. (19) ensures directly $C^1$ continuity because each of the functions $(11)$ in expression $(15)$ meets plane $\eta \xi$ in angle $\pi/2$. This is very important in the cases when the location of the artificial boundary is close to the contact area between the structure and the near field.

6. Conformal mapping into the infinite direction

Several types of conformal mapping can be used to map the infinite size of the local coordinate $\xi \in [0, \infty)$ to finite size $\xi \in [-1, 1]$ or $\xi \in [0,L]$. There are two main approaches – with mid nodes and without mid nodes. The first is described in [19] and can be explained directly by geometrical mapping of local to global coordinates. This approach maps $\xi$ into $\xi \in [-1, 1]$ and is very effective.

7. Mapped infinite element with mid nodes

In Fig. 3 the principles of generation of the derived mapping functions inside the infinite element with mid nodes and the boundary between finite and infinite elements are illustrated.

![Fig. 3. Mapping infinite to finite domain](image-url)
Consider the following mapping function similar to expression (4):

\[ x = -\frac{\xi}{1-\xi} x_R + \left(1 + \frac{\xi}{1-\xi}\right) x_Q. \]  

(20)

It is obviously also that when

\[ \xi = 0 \rightarrow x = x_R, \]
\[ \xi = 1 \rightarrow x = \infty, \]
\[ \xi = -1 \rightarrow x = \frac{x_R + x_Q}{2} = x_Q. \]  

(21)

Finally the mapping can be expressed as:

\[ x = \left(1 + \frac{2\xi}{1-\xi}\right) x_R - \frac{2\xi}{1-\xi} x_Q = N_R x_R + N_Q x_Q. \]  

(22)

8. Mapping infinite element without mid nodes

In this paper a new approach with no-mid nodes to mapping the local coordinate \( \xi \in [0, \infty) \) to finite size \( \xi \in [0, L] \) is proposed. After that, the geometrical mapping of local to global coordinates can be obtained directly. This approach is based on the conformal mapping such as:

\[ \xi = L \sec h \bar{\xi} = L \frac{1}{\cosh \bar{\xi}}, \]  

(23)

for the shape function \( N_{iq}^{fd}(x,z,\omega) = L_q (\eta) W_q^{fd}(\bar{\xi},\omega), \) and

\[ \xi = L \tanh \bar{\xi} = L \frac{\sinh \bar{\xi}}{\cosh \bar{\xi}}, \]  

(24)

for the shape function \( N_{iq}^{fd}(x,z,\omega) = L_q (\eta) W_q^{fd}(\bar{\xi},\omega). \)

Another possible conformal mapping form for the shape function (19) is:

\[ \xi = L \frac{1}{1 + \xi}, \]  

(25)

where \( s \) can be chosen as function of the attenuation factor \( \alpha \) as:

\[ s = f(\alpha). \]  

(26)

Following the same steps as in [27], the wave propagation functions included in normalized superposed shape function, can be given as:

- For \( p \) waves in the nodes on the line \( \bar{\xi} = 0 \)

\[ \text{Re} W(\xi) = \frac{1}{m} \sum_{q=1}^{m} \sum_{n=-\infty}^{\infty} 1 \cos \left(1 \frac{\text{int}}{L \ c_p} \sec h^{-1} \bar{\xi} \right) \ e^{-\frac{1}{L} \ c_p \ \text{sec h}^{-1} \bar{\xi}} \]  

or

(27)
• for \( p \) waves in the nodes on the line \( \xi = \infty \)

\[
\text{Re} W(\xi) = \frac{1}{m} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \left( \frac{1}{L c_p} \tanh^{-1} \xi \right) e^{-\frac{1}{L} \alpha \tanh^{-1} \xi} \quad \text{and} \quad (28)
\]

• For \( s \) waves in the nodes on the line \( \xi = 0 \)

\[
\text{Re} W(\xi) = \frac{1}{m} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \left( \frac{1}{L c_s} \sec^{-1} \xi \right) e^{-\frac{1}{L} \alpha \sec^{-1} \xi} \quad \text{or} \quad (29)
\]

• for \( s \) waves in the nodes on the line \( \xi = \infty \)

\[
\text{Re} W(\xi) = \frac{1}{m} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos \left( \frac{1}{L c_s} \tanh^{-1} \xi \right) e^{-\frac{1}{L} \alpha \tanh^{-1} \xi} \quad , \quad (30)
\]

using finite number of expressions for the individual wave propagation functions [25].

9. Hyperbolic mapping remarks

The hyperbolic mapping of the infinite direction to finite direction (\( \xi \leftrightarrow \bar{\xi} \)) in the proposed variant of an elastodynamical infinite element has many advantages. The first and very important advantage is that many of the properties of the trigonometric functions are valid and the second one is that the implementation in the standard finite element method is easier.

Consider the mapping given by (24) in the case \( L=1 \)

\[
\xi = \tanh \bar{\xi} = \frac{\sinh \bar{\xi}}{\cosh \bar{\xi}} . \quad (31)
\]

Equation (31) can be written as:

\[
\xi = \tanh \bar{\xi} = e^{\bar{\xi}} - e^{-\bar{\xi}} = \frac{e^{2\bar{\xi}} - 1}{e^{2\bar{\xi}} + 1} , \quad (32)
\]

and also in Taylor series such as:

\[
\xi = \tanh \bar{\xi} = \bar{\xi} - \frac{1}{3} \bar{\xi}^3 + \frac{2}{15} \bar{\xi}^5 - \frac{17}{315} \bar{\xi}^7 + \frac{62}{2835} \bar{\xi}^9 ... \quad (33)
\]

10. Conclusion

This paper presents a variant of an elastodynamical infinite element, appropriate for an adequate simulation of soil-structure interaction problems. This element is a new form of the infinite element proposed recently in [27]. By means of new conformal mapping the infinite domain of the element is mapped into a finite one. New forms of wave propagation
functions are obtained. The study is based on horizontal type (HIE) elements. With similar techniques vertical (VIE) and corner (CIE) elements can be formulated. Shape functions are divided into two types. The first is referred to the nodes on the line $\xi = 0$ or $x_0$ line in the global coordinate system and the second is referred to the nodes on the line $\xi = \infty$.

One of the important features of this element is that we do not need the conformal mapping because we are generally interested in the near field. Once the standard matrix has been obtained we do not need the inverse transformation. So the implementation is not difficult to be made.

This class of infinite elements are much similar to finite elements. The advantage of each mapping form is that the element computations can be done using standard computational strategy for the finite element method.

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