Review of Plane Stress and Plane Strain Elasticity

1. 3D Elasticity Problem

The three dimensional problem of the theory of elasticity (3D elasticity problem) consists of:

- Governing differential equation (equilibrium equations) + boundary conditions
- Strain-displacement relationship (Kinematic equations, Cauchy equations or equations of the geometry)
- Stress-strain relationship (Constitutive equations)

There are some special cases:

- 2D (plane stress, plane strain)
- Axisymmetric body with axisymmetric loading
- Principle of minimum potential energy

This lecture reviews aspects of structural mechanics within the following assumptions:

- Small displacements — equilibrium in the displaced configuration of a structure is adequately represented by the geometry of the original configuration.
- Small strains — second order derivatives of strain have small effect.
- Linear, elastic materials — the material is elastic (all strain energy is recoverable) and the stress is a linear function of strain.

The equations of elasticity relate strain and stress in such a way that equilibrium, compatibility, and the elastic material law are all satisfied. Later in the course we will relax some of these assumptions and consider second-order deformations and, possibly, inelastic materials.

The following figure shows 3D Elasticity Problem definition
where

\( V \): Volume of body

\( S \): Total surface of the body

The deformation at point \( x = [x \ y \ z]^T \) is given by the 3 components of its displacement

\[
\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}
\]

We have to note that \( \mathbf{u} = \mathbf{u}(x, y, z) \), i.e., each displacement component is a function of position.

Two basic types of external forces act on a body

1. **Body force** (force per unit volume) e.g., weight, inertia, etc
2. **Surface traction** (force per unit surface area) e.g., friction

The body forces are shown in the Figure:
The forces are as follows:

**Body force:** distributed *force per unit volume* (e.g., weight, inertia, etc). It is denoted

\[
g = \begin{bmatrix}
x \\
y \\
 Z
\end{bmatrix}
\]

**NOTE:** If the body is accelerating, then the *inertia force*

\[
-\rho \ddot{u} = \begin{bmatrix}
-\rho \ddot{u} \\
-\rho \ddot{v} \\
-\rho \ddot{w}
\end{bmatrix}
\]

may be considered as a part of $g$.

The surface traction is shown in the Figure.
**Traction:** Distributed force per unit surface area and it is represented by the vector

\[
p = \begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix}
\]

The quantities that describe the stress and strain state of the structure are:

- \(\sigma_x, \sigma_y, \sigma_z\) – normal stresses
- \(\tau_{xy}, \tau_{yz}, \tau_{zx}\) – shear stresses
- \(\varepsilon_x, \varepsilon_y, \varepsilon_z\) – normal strains
- \(\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\) – shear strains

**2. Plane Stress Problem**

**Assumptions:**
1. \(t \ll D\)
2. Top and bottom surfaces are free from traction, i.e., $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

**Plane stress examples**

1. **Thin plate with a hole**

2. **Thin cantilever plate**
The nonzero stresses are:

\[ \sigma_x = \sigma_x(x, y), \]

\[ \sigma_y = \sigma_y(x, y), \]

\[ \tau_{xy} = \tau_{xy}(x, y). \]

3. Plane Stain Problem

Assumptions:

- The length of the structure is very large in comparison with the other two dimensions
- The loads are applied only normally to the longitudinal axis (the loads are perpendicular to the \( z \)-axis. The loads are uniformly distributed along the length of the body.
- The support conditions are the same along the \( z \)-axis

When the structure satisfies the above assumptions all slices has the same stress and strain state. The we can write

\[ u = u(x, y), \]
\[ v = v(x, y), \]
\[ w = 0. \]
3. Basic Equations for Plane Stress and Plane Strain

3.1. Equilibrium

The equilibrium equation is a differential equation in the deformed configuration. If displacements are small, the equilibrium relationship is valid in the original configuration. More advanced theories define stress tensors for large displacements.

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0.
\]

3.2. Kinematic Relationships

The displacements relate the change in geometry from the original to the displaced configuration. Again, we are assuming displacements much smaller than the dimensions of the system. The strain components are defined as

\[
\varepsilon_x = \frac{\partial u}{\partial x}, \\
\varepsilon_y = \frac{\partial v}{\partial y}, \\
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.
\]

3.3. Constitutive Relationship for Linear Elastic Materials

An elastic material is one in which the stress is a unique function of strain, independent of how the strain is achieved. Another way of saying this is that there is a strain energy function that only depends on the value of the stress and strain tensor. If the stress is a linear function of strain, then Hooke’s law is valid.

We consider homogeneous and isotropic material. For homogeneous material, the properties are the same at any point of the body. For an isotropic material, the properties are the same in any direction. There are two independent elastic constants using the symmetry of an isotropic material. A common choice for the isotropic constants are:

- \( E \) — Young’s modulus of elasticity
- \( \nu \) — Poisson’s ratio

The strain–displacement relationship for an isotropic material can be written in terms of \( E \), and \( \nu \) is:
\[ \sigma_x = \frac{E}{1-\nu^2} \left( \varepsilon_x + \nu \varepsilon_y \right), \]
\[ \sigma_y = \frac{E}{1-\nu^2} \left( \varepsilon_y + \nu \varepsilon_x \right), \]
\[ \tau_{xy} = G \gamma_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} = \frac{E(1-\nu)}{2(1+\nu)(1-\nu)} \gamma_{xy} = \frac{E}{1-\nu^2} \cdot \frac{1-\nu}{2} \gamma_{xy}, \]

where \( G \) is a shear modulus.

For the plane strain the moduli \( E \) and \( \nu \) should be substituted with the following reduced moduli
\[ E_1 = \frac{E}{1-\nu^2} \quad \text{and} \quad \nu = \frac{\nu}{1-\nu}, \]
respectively.

### 3.4. Boundary conditions

The governing equations are complete except for boundary conditions. There are two types of conditions at the boundary of a structure:

**Displacement boundary conditions**

\[ u = u_0, \quad v = v_0 \]

at the part of boundary \( S_u \)

and

**Traction boundary conditions**

\[ X_n = \sigma_x \cos (n, x) + \tau_{nx} \cos (n, y) \]
\[ Y_n = \tau_{xy} \cos (n, x) + \sigma_y \cos (n, y) \]

Each part of the boundary must have a displacement specified or traction in each direction.

### 3.5. Matrix Form of the Governing Equations

Let us introduce the following vectors and matrices
\[ \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \quad u = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} \]
\[
[\varepsilon] = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
\]
\[g = \begin{bmatrix} X \\ Y \end{bmatrix} \quad p = \begin{bmatrix} X_n \\ Y_n \end{bmatrix}\]

Additionally we define initial strain and initial stresses \( \varepsilon_0 \) and \( \sigma_0 \).

For example initial strain vector due to the temperature variation are
\[
\varepsilon_0 = \begin{bmatrix} \alpha \Delta t \\ \alpha \Delta t \\ 0 \end{bmatrix} \quad \text{and} \quad \varepsilon_0 = (1 + \nu) \begin{bmatrix} \alpha \Delta t \\ \alpha \Delta t \\ 0 \end{bmatrix}
\]
for the plane stress and plane strain, respectively.

We can now write the governing equations in the matrix form as follows:

**Equilibrium**
\[ [\varepsilon]^T \sigma + g = 0 \]

**Kinematic relationships (Cauchy)**
\[ \varepsilon = [\varepsilon] u \]

**Hooke’s law**
\[ \sigma = D(\varepsilon - \varepsilon_0) + \sigma_0 \]

**Displacement boundary conditions**
\[ u = u_0 \quad \text{at boundary} \ S_u \]